



Stabilizability Analysis of Multiple Model Control with Probabilistic Switching

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ABSTRACT

In this paper, we derive some useful necessary conditions for stabilizability of multiple model control using a bank of stabilizing state feedback controllers. The outputs of this set are weighted by their probabilities as a soft switching system and together fed back to the plant. We study quadratic stabilizability of this closed loop soft switching system for both continuous and discrete-time hybrid system. For the continuous-time hybrid system, a bound on sum of eigenvalues of P_i is found when their derivatives of Lyapunov functions are upper bounded. For discrete-time hybrid system, a new stabilizability condition of soft switching signals is presented.

Keywords: Hybrid Multiple Models Control, Lyapunov Function, Soft Switching System.

1. INTRODUCTION

A switched linear system is hybrid dynamical system which consists of several linear subsystems and a switching rule that decides which of switching rule is active at each moment. In the last two decades, there has been increasing interest in stability analysis and control design for switched systems in [1], [2], [3], [4], [5], [6], [7], [8], [9], [15], [16], [17], [18], and [19]. The motivation for studying switched systems is from the fact that many practical systems are inherently multi-modal. Many researchers have studied the use of multiple models in adaptive control of both linear and nonlinear in which controllers are switched depending on which model provides the least identification errors. Stability results for such continuous time, switching control systems have been shown for the linear [10] and for a certain nonlinear case [11]. The linear multiple model switching is similar to the control of Markovian jump linear systems or the system has models whose parameters change with respect to an underlying Markov chain. This paper considers the problem of adaptive control of multiple models. However, the study is not concerned with instantaneous switches among models referred to as a hard switching system, but instead with a probability-based weighting of outputs of different models referred to as a soft switching system.

A simplest continuous time hybrid system is described by the following different linear state update equation:

$$\dot{x}(t) = A_i x(t) + B_i u(t) \quad (1)$$

$$z(t) = C_i x(t) \quad (2)$$

And a discrete-time hybrid system is the following:

$$x(k+1) = A_i x(k) + B_i u(k) \quad (3)$$

$$z(k) = C_i x(k) \quad (4)$$

in which $[A_i, B_i, C_i]$ are the time varying state space model matrices, $i = \{1, 2, \dots, N\}$, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $z \in \mathbb{R}^p$ is the measured output.

When the model uncertainty is present, the exact plant model $[A, B, C]$ is unknown. The model uncertainty is described by $\Omega: [A, B, C] \in \Omega = Co\{[A_1, B_1, C_1], \dots, [A_N, B_N, C_N]\}$, a convex hull of $[A, B, C] = \sum_{i=1}^N \mu_i [A_i, B_i, C_i]$. The weighting factors μ_i denotes probability: $\sum_{i=1}^N \mu_i = 1$, $\mu_i \geq 0$. The transition probability $\mu_{ij}(t)$, i.e. the probability that the system will jump from mode m_i to mode m_j at time instant t , is assumed to be a first order Markov chain.

The exact plant model $[A, B, C]$ can be detected by using algorithms of multiple model estimators. A bank of filters runs in parallel every time, each based on a particular model, to obtain the model-conditional estimates $\hat{x}_i(t)$ and the likelihood probability $\mu_i(t)$ of each model $[A_i, B_i, C_i]$ matching to the exact plant. The outputs of this set are weighted by their probabilities and together fed back to the plant. Methods that employ such approaches are called Interacting Multiple Model (IMM) estimation techniques [12]. These methods are not discussed in this paper.

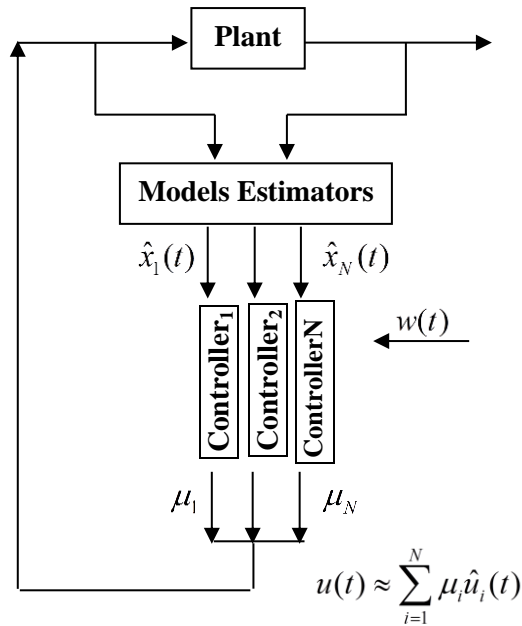


Figure 1. Diagram of Model Estimators based feedback Controllers

For the controller reconfiguration, we can apply hard switching or soft switching. For hard switching, we use only one controller implemented at any time. IMM estimator provides the overall state estimate $\hat{x}(t)$ and indicates one “most reliable” model $[A_i, B_i, C_i]$ in the model set Ω . Thus, we can build up a stabilizing controller corresponding to this “most reliable” model. The hard switching system seems unrealistic since the exact plant model is varying in a convex hull of

$[A, B, C] = \sum_{i=1}^N \mu_i [A_i, B_i, C_i]$. Hence, we consider the use of soft switching signals where the overall control input is the convex combination of several stabilizing state feed-back controllers $u(t) = \sum_{i=1}^N \mu_i \hat{u}_i(t)$ and switches among controllers are continuous and smooth based the model probabilities μ_i detected by the IMM estimator.

Next, we will investigate some necessary and sufficient conditions for stabilizing these soft switching signals.

2. STABILIZABILITY OF SOFT SWITCHING FOR CONTINUOUS-TIME

Equations (1), (2), and (5) can be combined to obtain the closed loop state transition matrix $A - BK = \sum_{i=1}^N \mu_i A_i - \sum_{i=1}^N \mu_i B_i K_i = \sum_{i=1}^N \mu_i (A_i - B_i K_i) = \sum_{i=1}^N \mu_i A_{CLi}$. Therefore, we have the closed loop equations:

$$\dot{x}(t) = \sum_{i=1}^N \mu_i A_{CLi} x(t) \tag{6}$$

$$z(t) = Cx(t) \tag{7}$$

As indicated in [1], even if each of closed loop matrices A_{CLi} are globally stable with their eigenvalues being absolutely negative, there can exist a switching sequence that destabilizes the closed loop dynamics.

For all given stable closed loop state matrices A_{CLi} , the stability of the soft switching system is guaranteed if we can find out a common positive symmetric definite Lyapunov matrix P and positive symmetric definite matrices Q_i such that $A_{CLi}P + PA_{CLi}' = -Q_i, \forall i$. For a positive Lyapunov function $V(x) = x'Px$, we have always a negative time derivative $\dot{V}(x) < 0$, and the system is stable for any linear switching systems with any switching signal sequence. Since all closed loop state matrices A_{CLi} are stable and the closed loop state update equation (6) is

$$\dot{x}(t) = \sum_{i=1}^N \mu_i A_{CLi} x(t) : \tag{8}$$

$$\dot{V}(x) = \left(\sum_{i=1}^N \mu_i A_{CLi} x \right)' Px + x' P \left(\sum_{i=1}^N \mu_i A_{CLi} x \right) = \sum_{i=1}^N \mu_i x' (A_{CLi}P + PA_{CLi}') x = \sum_{i=1}^N \mu_i x' (-Q_i) x < 0$$

The existence of a direct common Lyapunov matrix $A_{CLi}P + PA_{CLi}' = -Q_i$ among stable matrices A_{CLi} and positive symmetric definite matrices Q_i can be searched with quadratic stability of polytopic systems or directly solved with LMIs.

Example 2.1. Given four stable closed loop state matrices $A_{CL1} = \begin{bmatrix} -0.2 & -0.5 \\ 0.3 & 0.1 \end{bmatrix}$, $A_{CL2} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$, $A_{CL3} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, and $A_{CL4} = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$. By searching with quadratic stability of polytopic systems, we find out a common Lyapunov matrix for all four these matrices: $P_a = \begin{bmatrix} 0.8928 & 0.4107 \\ 0.4107 & 1.5454 \end{bmatrix}$. By solving directly with LMIs, we can find out

another common Lyapunov matrix: $P_b = \begin{bmatrix} 0.71311 & 0.2920 \\ 0.2920 & 1.0851 \end{bmatrix}$. And here, we can

conclude that the switching linear system for the above four stable closed loop state matrices are stable with any linear switching system and with any switching sequence.

The conditions for the existence of a common Lyapunov function is not easy to determine at present time. And generally, we cannot always find out a common Lyapunov matrix among stable closed loop state matrices A_{CLi} . In this example, if we

only change A_{CL4} by a new stable closed loop state matrix $A_{CL4}^{New} = \begin{bmatrix} -0.25 & 0.5 \\ -1 & 0.1 \end{bmatrix}$, there

is no solution for a common Lyapunov matrix among the above four stable matrices.

The existence for a common Lyapunov function is also derived from [13] and involves communication relations among the A_{CLi} matrices. Because of the communication, it is easily to derive that $A_{CLi}A_{CLj} = A_{CLj}A_{CLi} \rightarrow e^{A_{CLi}t_1 A_{CLj}t_2} = e^{A_{CLj}t_2 A_{CLi}t_1}$. By direct calculation, it is straight forwards to verify that $\dot{V}(x) < 0$ and the switching system is stable if and only if all the closed loop state matrices A_{CLi} are stable and commute pairwise (i.e. $A_{CLi}A_{CLj} = A_{CLj}A_{CLi}$, $\forall i, j$).

The existence of a common Lyapunov function, although sufficient, is not necessary for stability of the switching systems. In this paper, we investigate some necessary conditions for stabilizability of the closed loop soft switching systems with an assumption that their derivatives of the Lyapunov functions is upper bounded. Thus, we can always find out positive constant scalars η_i for each model that

$$\dot{V}(x) = \sum_{i=1}^N \mu_i \dot{V}_i(x) = \left(\sum_{i=1}^N \mu_i A_{CLi} x \right)' P_i x + x' P_i \left(\sum_{i=1}^N \mu_i A_{CLi} x \right) \leq -\eta_i(x'x), \quad \forall \mu_i \quad (9)$$

And now a natural question arises that under what conditions, the equation (9) provides solution for all positive definite matrices P_i . The answer for the question can be found in the following theorem which gives necessary conditions to the existence of positive definite matrices P_i to the equation (9).

The following notation is used, $tr(P)$, $\lambda_{\min}(P)$, $\lambda_{\max}(P)$, denote the trace, the minimum eigenvalue, the maximum eigenvalue of the matrix P . And $\eta_{\min} = \min_i(\eta_i)$. A new upper bound for the sum of eigenvalues of P_i is found and the switched systems are global asymptotical stability.

Theorem 1: Consider the continuous time, stable closed loop state system described by equations (6) and (7) and the derivatives of the Lyapunov functions are assumed to be upper bounded as in equation (9). Suppose that $P_i > 0$ is the Lyapunov matrices for each stable closed loop mode A_{CLi} . The sum of the eigenvalues of all P_i has an upper

$$\text{bound: } tr(P_i) \leq -\frac{\eta_{\min}}{2\lambda_{\min}(S_{CLi})}, \text{ where } S_{CLi} = \frac{A_{CLi} + A_{CLi}'}{2} \text{ and } \lambda_{\min}(S_{CLi}) < 0.$$

Proof. Since the derivatives of the Lyapunov functions are assumed to be upper bounded from equation (9), we can write

$$\sum_{i=1}^N \mu_i (A_{CLi} P_i + P_i A_{CLi}') \leq -\eta_i I \quad (10)$$

Or

$$\sum_{i=1}^N \mu_i \left(\text{tr}(A_{CLi}P_i) + \text{tr}(P_i A'_{CLi}) \right) \leq -\eta_{\min} \quad (11)$$

Using the matrix trace property $\text{tr}(AP) = \text{tr}(PA)$, we obtain

$$\text{tr}(A_{CLi}P_i) + \text{tr}(P_i A'_{CLi}) = 2\text{tr}(P_i S_{CLi}) \quad \text{where } S_{CLi} = \frac{A_{CLi} + A'_{CLi}}{2} \quad (12)$$

Considering the following inequality in [14]

$$\lambda_{\min}(S)\text{tr}(P) \leq \text{tr}(PS) \leq \lambda_{\max}(S)\text{tr}(P) \quad (13)$$

Using (12) and (13) in (11) yields

$$\text{tr}(P_i) \leq -\frac{\eta_{\min}}{2\lambda_{\min}(S_{CLi})} \quad (14)$$

Given the difficulty in finding the positive scalars η_i , we can search with LMIs software for the upper bounds on time derivatives of $V_i(x)$ for each model. Once the η_{\min} is known, theorem 1 is held.

3. STABILIZABILITY OF SOFT SWITCHING FOR DISCRETE-TIME

Equations (3), (4) and (5) can be combined to obtain the closed loop state transition matrix $A - BK = \sum_{i=1}^N \mu_i A_i - \sum_{i=1}^N \mu_i B_i K_i = \sum_{i=1}^N \mu_i (A_i - B_i K_i) = \sum_{i=1}^N \mu_i A_{CLi}$. Therefore, we have the closed loop equations:

$$x(k+1) = \sum_{i=1}^N \mu_i A_{CLi} x(k) \quad (15)$$

$$z(k) = Cx(k) \quad (16)$$

As indicated in [1], even if each of matrices A_{CLi} are globally stable with their eigenvalues $0 < \lambda_{A_{CLi}} < 1$, there can exist a switching sequence that destabilizes the closed loop dynamics. For all given stable matrices A_{CLi} , the stability of the switching systems is guaranteed if we can find out a common Lyapunov matrix P .

Lemma 3.1: *The switched linear systems for stable closed loop uncertainties A_{CLi} can guarantee the global asymptotical stability for any switched linear systems with any switching signal sequence if there exists a common positive symmetric definite matrix*

$$P > 0 \quad \text{and a scalar } \gamma > 0 \quad \text{such that} \quad \begin{bmatrix} P & PA'_{CLi} & \gamma \\ A_{CLi}P & P & 0 \\ \gamma & 0 & \gamma I \end{bmatrix} > 0, \quad \forall i.$$

Proof: For the stable discrete time systems, we always have the Lyapunov function decreasing $V_i(k) = x'(k)Px(k)$ and $V_i(k+1) - V_i(k) < 0$, and the system is stable for any switched systems with any switching signal sequence since they share a common Lyapunov matrix P :

$$V_i(k+1) - V_i(k) = \left(\sum_{i=1}^N \mu_i A_{CLi} x \right)' P \left(\sum_{i=1}^N \mu_i A_i x \right) - xPx < 0 \rightarrow A_{CLi}PA'_{CLi} - P < 0 \quad (17)$$

By adding a scalar $\gamma > 0$ in equation (17), we have $P - A'_{CLi}PA_{CLi} - \gamma I > 0$, or $P - (A'_{CLi}P)P^{-1}(PA_{CLi}) - (\gamma)I\gamma^{-1}(\gamma) > 0$. And using Schur complement, this equation is equivalent to the LMI in lemma 3.1.

Hence, the common Lyapunov matrix in lemma 3.1 is the solution to the following

$$\text{LMI: } \min_{P>0, \gamma>0} \gamma, \text{ subject to } \begin{bmatrix} P & PA'_{CLi} & \gamma \\ A_{CLi}P & P & 0 \\ \gamma & 0 & \gamma I \end{bmatrix} > 0, \forall i.$$

The difference $V(k+1) - V(k)$ in equation (17) involves many cross matrices of form $A_{CLi}PA'_{CLi} - P < 0$. In this paper we do not analyse the necessary conditions that guarantee the stability of discrete closed loop switching signals. Instead, we look at $A_{CLi}A'_{CLi}$. If $\lambda_{\max}(A_{CLi}A'_{CLi}) < 1$, the closed loop system is strictly stable since we always have $\frac{x'(A_{CLi}A'_{CLi})x}{x'x} \leq \lambda_{\max}(A_{CLi}A'_{CLi})$. Then,

$$V_i(k+1) - V_i(k) = x'(A_{CLi}PA'_{CLi})x - x'Px \rightarrow \lambda_{\max}(A_{CLi}A'_{CLi}) - I < 0 \quad (18)$$

And the closed loop state feedback A_{CLi} is strictly stable. Now we use the above notion to prove a new stabilizability condition of soft switching signals for discrete-time case.

Theorem 2: *The switched linear systems for the discrete-time hybrid system in (15) and (16) can guarantee the global asymptotical stability for any switched linear systems with any switching signal sequence if all $\lambda_{\max}(A_{CLi}A'_{CLi}) < 1, \forall i$.*

Proof: We have

$$\dot{V}(x) = \sum_{i=1}^N \mu_i \dot{V}_i(x) = \sum_{i=1}^N \mu_i (V_i(k+1) - V_i(k)) = \sum_{i=1}^N \mu_i (x'(A_{CLi}P_iA'_{CLi})x - x'P_i x) < 0 \quad (19)$$

For all given stable matrices A_{CLi} , the stability of the switching systems is guaranteed if we can find out a common Lyapunov matrix P . Here we have selected the common Lyapunov matrix $P = I$. Equation (19) can be transformed as

$$\frac{1}{x'x} \dot{V}(x) = \sum_{i=1}^N \mu_i \left(\frac{x'(A_{CLi}A'_{CLi})x}{x'x} - I \right) < 0 \quad (20)$$

Since we always have $\frac{x'(A_{CLi}A'_{CLi})x}{x'x} \leq \lambda_{\max}(A_{CLi}A'_{CLi}) < 1$. Then equation (20) is always held and the switching systems are global asymptotical stability for any switched linear systems with any switching signal sequence.

4. CONCLUSIONS

In this paper, we have considered the stability and stabilizability of soft switching systems for polytopic uncertainties via their stable closed loop state feedback controllers. For the continuous-time hybrid system, a bound on sum of eigenvalues of P_i is found when their derivatives of Lyapunov functions are upper bounded. For discrete-time hybrid system, a new stabilizability condition of soft switching signals is presented.

There are several important issues which should be studied in the future work. Firstly, we just highlight some necessary conditions for stabilizability of the closed loop state feedback but we still do not provide general stabilizing switching laws for hybrid multiple model control. Secondly, we have not considered the hybrid multiple models control subject to constraints and disturbances.

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CONFLICT OF INTERESTS

The author would like to confirm that there is no conflict of interests associated with this publication and there is no financial fund for this work that can affect the research outcomes.

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